THE WEAK STOCHASTIC REALIZATION PROBLEM FOR DISCRETE-TIME COUNTING PROCESSES

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Abstract. The weak stochastic realization problem is considered for discrete-time stationary counting processes. Such processes take values in the countable infinite set $N = \{0,1,2,\ldots\}$. A stochastic realization is sought in the class of stochastic systems specified by a conditional distribution for the output given the state of Poisson type, and by a finite valued state process. In the paper a necessary and sufficient condition is derived for the existence of a stochastic realization in the above specified class.

1. INTRODUCTION

The purpose of this paper is to present a result for the weak stochastic realization of a discrete-time counting process and to indicate the major open questions.

The weak stochastic realization problem to be considered is given a discrete-time counting process to show existence of and to classify all minimal Poisson-finite-state stochastic systems whose output equals the given process in distribution. The class of Poisson-finite-state stochastic systems is specified by a conditional distribution for the output given the state of Poisson type, and by a finite valued state process.

The motivation of this problem is the area of control and prediction for systems with point process observations. Examples of practical problems in this area are the control of queues, the prediction of traffic intensities, the estimation of software reliability, and the estimation of certain biomedical signals. The prediction and control problems for this class of systems, under the assumption that the parameter values are known, have been considered. Practical application of these results demands the solution of the system identification problem and the stochastic realization problem for the class of Poisson-finite-state systems.

The stochastic realization problem for Gaussian processes has received quite some attention the past fifteen years [2,3,6]. Both the weak and the strong version of the problem have been investigated. A considerable body of results is available for this problem. The corresponding problem for finite valued processes for which a realization is sought in the class of stochastic systems with a finite state process has also received consideration [4,5,8]. However, little progress has been made on this problem as far as a realization algorithm and the characterization of minimal realizations is concerned. The major bottle neck is a factorization question for nonnegative matrices [5].

In this paper attention is focused on the weak stochastic realization problem for stochastic processes taking values in the positive integers. This problem should be distinguished from the finite stochastic realization problem for processes taking values in a finite set. A weak stochastic realization is sought in the class of Poisson-finite-state stochastic systems described above. A necessary and sufficient condition will be stated for a discrete-time counting process to have a realization in this class. Open questions will be mentioned.

A summary of the paper follows. The problem formulation is given in section 2, while in section 3 a condition for existence of a weak stochastic realization is derived.

2. PROBLEM FORMULATION

Below a definition is given of a Poisson-finite-state stochastic system and the corresponding weak stochastic realization problem is formulated.

Notation and terminology that will be used in the paper, will be defined. Let $\{\Omega,F,P\}$ be a complete probability space and $T = Z$ be the time index set. The conditional independence relation for a triple of σ-algebra's F_1, F_2, G is defined by the condition that

$$E[x_1 x_2 | G] = E[x_1 | G] E[x_2 | G]$$

for all $x_1 \in L^+(F_1)$ and $x_2 \in L^+(F_2)$; notation $(F_1, G, F_2) \in CI$. Here $L^+(F_1)$ is the set of all positive F_1 measurable random variables. The smallest σ-algebra with respect to which a random variable x is measurable is denoted by F^x, and that containing the σ-algebra's G and H by $G \vee H$. The set of positive integers is denoted by $N = \{0,1,2,\ldots\}$, while that of strictly positive integers by $Z_+ = \{1,2,3,\ldots\}$. For $n \in Z_+$ is $Z_n = \{1,2,\ldots,n\}$. The set of nonnegative matrices is denoted by $R_+^{n\times n}$. For material on this set see [1].

<u>2.1. DEFINITION.</u> A *Poisson-finite-state stochastic system* is a collection

$$\sigma = \{\Omega,F,P,T,N,B_N,X,B_X,n,\lambda\}$$

where $\{\Omega,F,P\}$ is a complete probability space, $T = Z$, $N = \{0,1,2,\ldots\}$, $X = \{c_1,c_2,\ldots,c_n\} \subset (0,\infty)$ for some $n \in Z_+$, B_N, B_X are σ-algebra's on N and X generated by all subsets of N and X, $n: \Omega \times T \to X$, $\lambda: \Omega\times T \to X$ are stochastic processes called respectively the *output process* and the *state process*, such that for all $t \in T$, $k \in N$

$$E[I_{(n_t=k)} | F_{t-1}^{n-} \vee F_\infty^\lambda] = (\lambda_t)^k \exp(-\lambda_t)/k!$$

and $(\lambda_t, F^{n-}_{t-1} \vee F^{\lambda-}_t,\ t \in T)$ is a stationary finite-state Markov process. Here $F^{n-}_t = \sigma(\{n_s,\ \forall\ s \le t\})$, $F^{\lambda}_{\infty} = \sigma(\{\lambda_s,\ \forall\ s \in T\})$.
Notation: $\sigma \in$ PFSΣ.

In a stochastic system one exhibits, besides the externally available output process, the underlying state process. The state process is of crucial importance for the solution of prediction and control problems. The above defined stochastic system is called Poisson-finite-state because the conditional distribution of the output process given the past and the state process is of Poisson type, and because the state process is a finite-state Markov process.

In the following a stochastic process taking values in N will be called a discrete-time counting process. The output of a Poisson-finite-state stochastic system is a discrete-time counting process.

An abstract definition of a stochastic system can also be given [4,5,8]. It can then be shown that the above defined Poisson-finite-state stochastic system satisfies this abstraction definition. For the sake of completeness this result is put on record.

2.2 DEFINITION. A (discrete-time) *stochastic system* is a collection

$$\sigma = \{\Omega, F, P, T, Y, B_Y, X, B_X, y, x\}$$

where $\{\Omega, F, P\}$ is a complete probability space, $T = Z$, Y, X are sets and B_Y, B_X σ-algebra's on Y respectively X, $y\colon \Omega\times T \to Y$, $x\colon \Omega\times T \to X$ are stochastic processes called respectively the *output process* and the *state process*, such that for all $t \in T$

$$\left(F^{y+}_t \vee F^{x+}_t,\ F^{x_t},\ F^{x-}_t \vee F^{y-}_{t-1}\right) \in \mathrm{CI},$$

where

$$F^{y+}_t = \sigma(\{y_s,\ \forall s \ge t\}).$$

2.3 PROPOSITION. *A Poisson-finite-state stochastic system as defined in* 2.1 *is a stochastic system as defined in* 2.2.

PROOF. Let $t \in T$, $k \in N$, $i \in Z_n$. Then

$$E\left[I_{(n_t=k)}\, I_{(\lambda_{t+1}=c_i)} \,|\, F^{n-}_{t-1} \vee F^{\lambda-}_t\right]$$

$$= E\left[I_{(\lambda_{t+1}=c_i)} E\left[I_{(n_t=k)} \,|\, F^{n-}_{t-1} \vee F^{\lambda-}_{\infty}\right] | F^{n-}_{t-1} \vee F^{\lambda-}_t\right]$$

$$= E\left[I_{(\lambda_{t+1}=c_i)} (\lambda_t)^k \exp(-\lambda_t)/k! \,|\, F^{n-}_{t-1} \vee F^{\lambda-}_t\right]$$

$$= E\left[I_{(\lambda_{t+1}=c_i)} \,|\, F^{\lambda_t}\right] (\lambda_t)^k \exp(-\lambda_t)/k!$$

by $(\lambda_t, F^{n-1}_{t-1} \vee F^{\lambda-}_t, t \in T)$ a Markov process,

$$= E\left[I_{(n_t=k)} I_{(\lambda_{t+1}=c_i)} | F^{\lambda}t \right].$$

A monotone class argument then gives that $(F^{n_t} \vee F^{\lambda_{t+1}}, F^{\lambda_t}, F^{n-}_{t-1} \vee F^{\lambda-}_t) \in CI$. An induction procedure and another monotone class argument then yields that

$$(F^{n+}_t \vee F^{\lambda+}_{t+1}, F^{\lambda t}, F^{n-}_{t-1} \vee F^{\lambda-}_t) \in CI,$$

from which the result is easily deduced.

For future use a dynamic representation of a Poisson-finite-state stochastic system is derived. Define $x: \Omega\times T \to R^n$ by $x_{it} = I_{(\lambda_t=c_i)}$, and $c \in R^n$ by

$$c^T = (c_1 \dots c_n).$$

For $c \in R^n$ define the diagonal matrix

$$D(c) = \text{diag}(c_1, \dots, c_n) \in R^{n\times n}$$

with on the diagonal entries of the vector c. Let $b \in R^n$, $b_i = \exp(-c_i)$. Then

$$\begin{aligned}
&(\lambda_t)^k \exp(-\lambda_t)/k! \\
&= \sum_{i=1}^{n} \exp(-c_i)(c_i)^k I_{(\lambda_t=c_i)}/k! \\
&= b^T D(c)^k x_t/k!
\end{aligned}$$

Let $A \in R^{n\times n}$ be the transition matrix of the stationary finite-state Markov process λ; thus

$$A_{ij} = P(\{x_{i,t+1}=1\} \cap \{x_{jt}=1\})/P(\{x_{jt}=1\})$$

if well defined and zero otherwise. Then

$$E[x_{t+1}|F^x_t] = Ax_t.$$

Define

$$\Delta m_{1t} = x_{t+1} - Ax_t$$

$$\Delta m_{2kt} = I_{(n_t=k)} - b^T D(c)^k x_t/k!$$

Then Δm_{1t}, Δm_{2kt} are martingale increments:

$$E[\Delta m_{1t}|F^{n-}_{t-1} \vee F^x_t] = 0,$$

$$E[\Delta m_{2kt} | F^{n-}_{t-1} \vee F^{x}_{t}] = 0.$$

One obtains thus the representation

$$\begin{cases} x_{t+1} = Ax_t + \Delta m_{1t}, \\ I_{(n_t=k)} = b^T D(c)^k x_t/k! + \Delta m_{2kt}. \end{cases}$$

2.4 PROBLEM. The *Poisson-finite-state weak stochastic realization problem* is, given a stationary discrete-time counting process on T=Z, to solve the following subproblems:

a. To give necessary and sufficient conditions for the existence of a Poisson-finite-state stochastic system σ such that the output process of this system equals the given process in distribution; if such a system exists then it is called a *weak stochastic realization* of the given process;
b. to classify all minimal weak stochastic realizations, where minimal refers to the number of elements in the state space.

One may pose the question why for discrete-time counting processes attention is restricted to the class of Poisson-finite-state stochastic systems? The answer is that for systems in this class the stochastic filtering problem can easily be solved. Such systems may therefore be used in applications. The system identification problem then demands the estimation of the parameters of the filter representation. To answer questions about the identifiability of the parameters, the weak stochastic realization problem must be resolved.

For the sake of reference the solution to the stochastic filtering problem for a Poisson-finite-state stochastic system is stated below. No reference in the literature is known for this result but its proof is elementary.

2.5 PROPOSITION. *Assume given a Poisson-finite-state stochastic system with the representation*

$$x_{t+1} = Ax_t + \Delta m_{1t},$$

$$I_{(n_t=k)} = b^T D(c)^k x_t/k! + \Delta m_{2kt},$$

as described above. The solution of the stochastic filtering problem for this system is given by

$$\hat{x}_t = E[x_t | F^{n-}_{t-1}],$$

$$\hat{x}_{t+1} = A\hat{x}_t + \sum_{k=0}^{\infty} A[D(\hat{x}_t) - \hat{x}_t\hat{x}_t^T]$$

$$(D(c)^k b/k!)[b^T D(c)^k \hat{x}_t/k!]^{-1} I_{(n_t=k)}$$

$$= \sum_{k=0}^{\infty} [AD(\hat{x}_t)D(c)^k b/k!][b^T D(c)^k \hat{x}_t/k!]^{-1} I_{(n_t=k)}.$$

PROOF. Omitted. □

The solution of the above filtering problem is readily implemented. If $b_k \in R_+^n$, $k \in N$, is defined as $b_k = D(c)^k b/k!$ then one has the recursion

$$b_{k+1} = D(c) b_k/(k+1), \ b_0 = b.$$

3. THE RESULT

Below a necessary and sufficient condition is given for a discrete-time counting process to have a weak stochastic realization in the class of Poisson-finite-state stochastic systems.

Some remarks on notation follow. The family of finite dimensional distributions of a stationary counting process n is denoted by, for any $m \in Z_+$,

$$p_m(t_1,\ldots,t_m,k_1,\ldots,k_m) = P(\{n_{t_1} = k_1,\ldots,n_{t_m} = k_m\})$$

where $t_1,\ldots,t_m \in T$, $t_m \le t_{m-1} \le \ldots \le t_1$, and $k_1,\ldots,k_m \in N$. Because the process is stationary p_m is dependent on the t_i's only through $t_1-t_2, t_2-t_3,\ldots,t_{m-1}-t_m$.

If $c, b \in R_+^n$ then $D(c)D(b) = D(b)d(c)$, while $D(c)b = D(b)c$. Let $u \in R^n, u^T = (1\ 1 \ldots 1)$. A stochastic matrix is an element $A \in R_+^{n\times n}$ such that $u^T A = u^T$. Note that if $x: \Omega\times T \to R^n$ is defined as in section 2 by $x_{it} = I_{(\lambda_t = c_i)}$, that then $(x_{it})^2 = x_{it}$, while for $i \neq j$, $x_{it}x_{jt} = 0$.

3.1 THEOREM. *Assume given a stationary discrete-time counting process on* $T = Z$, *say with finite-dimensional distribution, for* $m \in Z_+$,

$$p_m(t_1,\ldots,t_m,k_1,\ldots,t_m).$$

There exists a weak stochastic realization of this process in the class of Poisson-finite-state stochastic systems iff there exists a $n \in Z_+$, *a stochastic matrix* $A \in R_+^{n\times n}$, *and* $r,c \in (0,\infty)^n$, *such that if* $b \in (0,\infty)^n$, $b_i = \exp(-c_i)$, *then for any* $m \in Z_+$, $t_1,\ldots,t_m \in T$, $t_m < t_{m-1} < \ldots < t_1, k_1,\ldots,k_m \in N$ *one has*

$$\begin{aligned} &p_m(t_1,\ldots,t_m,k_1,\ldots,k_m) \\ &= u\, D(b)D(c)^{k_1} A^{t_1-t_2} D(b)D(c)^{k_2} A^{t_2-t_3} \ldots \\ &\qquad \ldots D(b)D(c)^{k_m} r/k_1!k_2!\ldots k_m! \end{aligned}$$

The above existence criterion is analogous to that of the existence of a finite stochastic realization as given in [4]. However, there conditional distributions are used, as where here unconditional distributions are preferred. Remarks on a realization algorithm are given below the proof.

PROOF. a ⇒ Assume there exists a weak stochastic realization say specified by the representation

$$x_{t+1} = Ax_t + \Delta m_t,$$

$$I_{(n_t=k)} = b^T D(c)^k c_t/k! + \Delta m_{2kt},$$

as discussed in section 2. Let $r = E(x_t)$. Then for $t_1 < t_2$

$$\begin{aligned} &E[x_{t_2} I_{(n_{t_1}=k)} | F^{n-}_{t_1-1} \vee F^x_{t_1}] \\ &= E[x_{t_2} E[I_{(n_{t_1}=k)} | F^{n-}_{t_1-1} \vee F^{x-}_\infty] | F^{n-}_{t_1-1} \vee F^x_{t_1}] \\ &= E[x_{t_2} x^T_{t_1} D(c)^k b/k! | F^{n-}_{t_1-1} \vee F^x_{t_1}] \\ &= A^{t_2-t_1} x_{t_1} x^T_{t_1} D(c)^k b/k! \\ &= A^{t_2-t_1} D(x_{t_1}) D(c)^k D(b) u/k! \\ &= A^{t_2-t_1} D(b) D(c)^k x_t/k!, \end{aligned}$$

$$\begin{aligned} &E[x_{t_2} I_{(n_{t_1}=k)}] \\ &= A^{t_2-t_1} D(b) D(c)^k r/k!, \end{aligned}$$

$$p_1(t_1,k) = E[I_{(n_{t_1}=k)}] = u^T D(b) D(c)^k r/k!$$

It will be shown by induction that for

$$t_m < t_{m-1} < \ldots < t_2 < t_1 < t_0, \; k_1,\ldots,k_m \in N$$

$$\begin{aligned} &E[x_{t_0} I_{(n_{t_1}=k_1)} \ldots I_{(n_{t_m}=k_m)}] \\ &= A^{t_0-t_1} D(b) D(c)^{k_1} \ldots A^{t_m-t_{m-1}} D(b) D(c)^{k_m} r/k_1! \ldots k_m! \;. \end{aligned}$$

By the above this holds for $m = 1$. Suppose it is true for $m - 1$. Then

$$\begin{aligned} &E[x_{t_0} I_{(n_{t_1}=k_1)} \ldots I_{(n_{t_m}=k_m)}] \\ &= E[E[x_{t_0} I_{(n_{t_1}=k_1)} | F^{n-}_{t_1-1} \vee F^{x-}_{t_1}] \ldots I_{(n_{t_m}=k_m)}] \\ &= A^{t_0-t_1} D(b) D(c)^{k_1}/k_1! E[x_{t_1} E[x_{t_1} I_{(n_{t_2}=k_2)} \ldots I_{(n_{t_m}=k_m)}] \\ &= A^{t_0-t_1} D(b) D(c)^{k_1} \ldots D(b) D(c)^{k_m} r/k_1! \ldots k_m!, \end{aligned}$$

$$\begin{aligned} &p_m(t_1,\ldots,t_m,k_1,\ldots,k_m) \\ &= E[I_{(n_{t_1}=k_1)} \ldots I_{(n_{t_m}=k_m)}] \end{aligned}$$

$$= u^T D(b)D(c)^{k_1} \ldots A^{t_{m-1}-t_m} D(b)D(c)^{k_m} r/k_1! \ldots k_m!$$

b. $\Leftarrow$ If the indicated factorization exists then one has $n \in Z_+$, $A \in R_+^{n\times n}$ a stochastic matrix, and $c \in (0,\infty)^n$. One can then construct a probability space and a Poisson-finite-state stochastic system on it and part a. of the proof then shows that

$$E[I_{(n_{t_1}=k_1)} \ldots I_{(n_{t_m}=k_m)}]$$
$$= u^T D(b)D(c)^{k_1} A^{t_1-t_2} \ldots D(b)D(c)^{k_m} r/k_1!k_2! \ldots k_m!$$
$$= p_m(t_1,\ldots,t_m,k_1,\ldots,k_m). \qquad \square$$

A major unsolved question for the stochastic realization problem under discussion is the construction of a realization algorithm. The following heuristic procedure may be considered.

1. Assume that the function $k!p_1(t,k)$, as function of $k \in N$, is a positive Bohl function meaning that there exists a $n \in Z_+$, $h,g \in R_+^n$, $F \in R_+^{n\times n}$ such that

$$k!p_1(t,k) = h^T F^k g.$$

Assume further that F can be chosen diagonal, say $F = D(c)$ with $c \in R_+^n$. Define $b,d \in R_+^n$ as $b_i = \exp(-c_i)$, $d_i = \exp(c_i)$. Then

$$k!p_1(t,k) = h^T D(c)^k g = u^T D(h)D(c)^k g$$
$$= u^T D(b)D(d)D(h)D(c)^k g$$
$$= u^T D(b)D(c)^k D(d)D(h) g$$
$$= b^T D(c)^k r,$$
$$1 = \Sigma_{k=0}^{\infty} p_1(t,k) = u^T r.$$

2. Determine a stochastic matrix $A \in R_+^{n\times n}$ such that for all $t_1,t_2 \in T$, $t_2 < t_1$, $k_1,k_2 \in N$,

$$k_1!k_2!p_2(t_1,k_2,k_1,k_2) = u^T D(b)D(c)^{k_1} A^{t_1-t_2} D(b)D(c)^{k_2} r.$$

Step 1 and 2 determine $n \in Z_+$, $c \in (0,\infty)^n$, $A \in R_+^{n\times n}$.

3. Check whether the condition of theorem 3.1 holds for any $m \in Z_+$.

A major difficulty with the above algorithm is that nothing is known about factorization of positive functions as in step 1 above. In addition little is known about the factorization in step 2 of positive functions with more then one countable infinite index. Analogous difficulties occur in the finite stochastic realization problem [4,5].

Another major unsolved question is the characterization of minimal realizations.

It seems that this question is also analogous to that of the finite stochastic realization problem, see [5]. There it is shown that this question leads to a factorization problem for nonnegative matrices. The latter problem is unsolved.

REFERENCES

[1] BERMAN, A. & R.J. PLEMMONS, *Nonnegative matrices in the mathematical sciences*, Academic Press, New York, 1979.

[2] FAURRE, P., M. CLERGET & F. GERMAIN, *Opérateurs rationnels positifs*, Dunod, Paris, 1979.

[3] LINDQUIST, A. & G. PICCI, *On the stochastic realization problem*, SIAM J. Control Optim., 17 (1979), pp. 365-389.

[4] PICCI, G., *On the internal structure of finite-state stochastic processes*, in: *Recent Developments in Variable Structure Systems, Economics, and Biology*, Proc. of a U.S. -Italy Seminar, Taormina, Sicily, 1977, Lecture Notes in Econ. and Mathematical Systems, volume 162, Springer-Verlag, Berlin, 1978, pp. 288-304.

[5] PICCI, G. & J.H. VAN SCHUPPEN, *On the weak finite stochastic realization problem*, Proc. Colloque ENST-CNET: *Développements récents dans le filtrage et le contrôle des processes aléatoires*, to appear; also report BW 184/83, Centre of Mathematics and Computer Science, Amsterdam, 1983.

[6] RUCKEBUSCH, G., *A state space approach to the stochastic realization problem*, Proc. 1978 Int. Symp. on Circuits and Systems, New York, IEEE, New York, 1978, pp. 972-977.

[7] SNYDER, D.L., *Random point processes*, J. Wiley & Sons, New York, 1975.

[8] VAN SCHUPPEN, J.H., *The strong finite stochastic realization problem-preliminary results*, in: *Analysis and Optimization of Systems*, A. Bensoussan, J.L. Lions (eds.), Lecture Notes in Control and Info. Sci., volume 44, Springer-Verlag, Berlin, 1982, pp. 179-190.